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SYNTHESIS OF THE OPTIMAL CONTROL OF DYNAMICAL STRUCTURES†

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The problem of synthesizing an optimal control by choosing the structure, in non-linear dynamical systems with a random structure, is formulated. One of the possible approaches to solving this problem is considered: it uses a method from the theory of the optimal control of systems with distributed parameters and enables one to construct the density vector of the distributions of the process under consideration for all states in such a way as to guarantee an optimum of the selected probability functional. An example is given to illustrate the practical possibilities of the approach. © 1999 Elsevier Science Ltd. All rights reserved.

On the basis of well-known classical results [1], the problem of the optimal control of dynamical systems with a random structure can be solved only by controlling the system itself (or its actual structure), not by selecting the structure of the system.

We formulate the problem as follows. Consider a non-linear dynamical system with a random structure, described in the general case [1] in the lth state by the following vector equation

$$\dot{\mathbf{Y}} = \mathbf{f}^{(l)}(\mathbf{Y}, t) + H^{(l)}(\mathbf{Y}, t)\mathbf{V}t, \quad \mathbf{Y}(t_0) = \mathbf{Y}_0, \quad l = 1, ..., s \tag{1}$$

where l is the number of the state (structure), $\mathbf{f}^{(l)}(\mathbf{Y}, t), H^{(l)}(\mathbf{Y}, t)$ are non-linear vector- and matrix-valued functions of the appropriate dimensions $n^{(l)} \leqslant N$ and $m^{(l)} \times n^{(l)}$, $N = \max(n^{(l)} = n^{(s)})$, $\mathbf{Y}(t)$ is the N-dimensional state vector in any structure and $\mathbf{V}_t$ is Gaussian white vector-noise of dimension $m^{(l)}$.

It is required to find a law $\nu(\mathbf{Y}, t)$ $(l = 1, 2, \ldots, s)$ for the transition from one structure to another so as to guarantee the optimum value over a given time interval $T = [t_0, t_k]$ of a certain functional J_0, defined on the set of probabilities and depending, in general, non-linearly on the distribution density $\rho(\mathbf{Y}, t)$ of the state vector $\mathbf{Y}$

$$J_0 = \iint_{T\mathbf{Y}_*} \Phi[\rho(\mathbf{Y}, t)]d\mathbf{Y}dt$$

where $\mathbf{Y}_*$ is the domain of definition of the argument $\mathbf{Y}$ in which the optimum is sought and Φ is a given non-linear analytic function.

With this form of the criterion J_0, a fairly wide range of the optimum criteria used in practical work is covered:

the maximum (minimum) of the probability that a vector $\mathbf{Y}$ exists in the domain $\mathbf{Y}_*$ in the given time interval: $\Phi(\rho) = \pm\rho$;

the minimum of the deviation of the unknown density ρ from a given value g: $\Phi(\rho) = (\rho - g)^2$, $\Phi(\rho) = |\rho - g|$, $\Phi(\rho) = -\rho \ln (g/\rho)$ (the Kulback criterion), etc.;

the maximum of the information about the state vector $\mathbf{Y}$ (or its minimum entropy): $\Phi(\rho) = \rho \ln \rho$, $\Phi(\rho) = \rho(\partial \ln \rho/\partial \mathbf{Y})^2$ (the Fisher criterion); and others.

For the functional J_0 to attain an optimum value, it is proposed to control the density $\rho(\mathbf{Y}, t)$ by choosing a suitable process structure $\mathbf{Y}$; this choice, in turn, depends—given the physical and technological characteristics of the structures—on the form of the law governing the change of structure. Thus, the problem may also be formulated as the synthesis of a stochastic process with given characteristics—in this case, an optimal transition law between the structures of the process. The most suitable choice of a vector defining the control of such structural transitions in what follows is the intensity vector of state transitions [1]

$$\nu(\mathbf{Y}, t) = |0 \ \nu_{12}...\nu_{1s} \ \nu_{21} \ 0 \ \nu_{23}...\nu_{2s} \ \nu_{31} \ \nu_{32} \ 0 \ \nu_{34}...\nu_{s(s-1)} \ 0|^T$$

where $\nu_{lr}(\mathbf{Y}, t)$ is the intensity of transitions from state l to state r; in order, for example, to avoid frequent

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state changes, it is assumed that the intensity vector is defined in such a way as to minimize the following quadratic form over a given time interval T for $Y \in Y_*$

$$\min \iint_{TY_*} \nu^T(\mathbf{Y},t)\nu(\mathbf{Y},t)d\mathbf{Y}dt$$

Since the vector ν contains zero components, we shall essentially search in what follows not for the vector ν itself but for a vector ν_0 such that $\nu = E_0\nu_0$, where ν_0 is the vector formed from ν by omitting the zero components, and E_0 is obtained from the identity matrix by adding zero rows so as to obtain the suitable zero elements of ν.

Then the criterion J to be minimized finally takes the form

$$J = \iint_{TY_*} \{\Phi[\rho(\mathbf{Y},t)] + \nu_0^T(\mathbf{Y},t)\nu_0(\mathbf{Y},t)\}d\mathbf{Y}dt \tag{2}$$

In turn, the distribution density of the process $\mathbf{Y}_t$ described by Eqs (1) is

$$\rho(\mathbf{Y},t) = \sum_{l=1}^{s} \omega(\mathbf{Y},l,t) = \sum_{l=1}^{s} \omega^{(l)}(\mathbf{Y},t)$$

where $\omega^{(l)}(\mathbf{Y}, t)$ is the distribution density of the extended vector $\| \begin{smallmatrix} \mathbf{Y} \\ l \end{smallmatrix} \|$ (l is the index of the state).

The most characteristic in applications is a continuous process $\mathbf{Y}$ in which the reconstructed values of the lth state are identical with the final values of the process of the rth state, that is, in which the final conditions for the existence of the rth (preceding) structure are the initial conditions for the lth (next) structure. For such a process (according to the classification proposed in [1]: "process with instantaneous complete reconstruction"), the conditional probability density of reconstruction of realizations is a δ-function [1, p. 64], so that no points of discontinuity appear when the equations for the densities of existence of the process in the corresponding structures are combined. In that case, the functions $\omega^{(l)}(\mathbf{Y}, t)$ are described by the following system of generalized Fokker–Planck–Kolmogorov (FPK) equations [1]

$$\frac{\partial\omega^{(l)}(\mathbf{Y},t)}{\partial t} = L[\omega^{(l)}(\mathbf{Y},t)] - \sum_{r=1}^{s} \nu_{lr}(\mathbf{Y},t)\omega^{(l)}(\mathbf{Y},t) + \sum_{r=1}^{s} \nu_{rl}(\mathbf{Y},t)\omega^{(r)}(\mathbf{Y},t), \quad l = 1,\ldots,s$$

where L is the FPK operator. Introducing vectors $\nu_0(\mathbf{Y}, t)$ and $\boldsymbol{\omega}(\mathbf{Y}, t) = \| \omega^{(1)}(\mathbf{Y}, t) \ldots \omega^{(s)}(\mathbf{Y}, t) \|^T$, we obtain the following general equation

$$\frac{\partial\boldsymbol{\omega}(\mathbf{Y},t)}{\partial t} = L[\boldsymbol{\omega}(\mathbf{Y},t)] - [\Omega[\boldsymbol{\omega}(\mathbf{Y},t)](E_s \otimes I_s) - \boldsymbol{\omega}^T(\mathbf{Y},t) \otimes E_s]E_0\nu_0(\mathbf{Y},t) \tag{3}$$

$$\Omega(\boldsymbol{\omega}) = \operatorname{diag}(\omega^{(1)}, \ldots, \omega^{(s)})$$

where E_0 is the $s \times s$ identity matrix, I_s is a row of ones of dimension s and $\otimes$ is the symbol of the Kronecker product.

Taking into consideration that, with the vector $\boldsymbol{\omega}$ defined as above, the expression for the density is $\rho(\mathbf{Y}, t) = I_s\boldsymbol{\omega}(\mathbf{Y}, t)$, we can express functional (2) as follows:

$$J = \iint_{TY_*} \{\Phi[I_s\boldsymbol{\omega}(\mathbf{Y},t)] + \nu_0^T(\mathbf{Y},t)\nu_0(\mathbf{Y},t)\}d\mathbf{Y}dt = \int_T W(t)dt \tag{4}$$

and, in order to simplify the subsequent solution, we rewrite Eqs (3) as follows:

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = L(\boldsymbol{\omega}) - [\Omega(\boldsymbol{\omega})(E_s \otimes I_s) - \boldsymbol{\omega}^T \otimes E_s]E_0\nu_0 = L(\boldsymbol{\omega}) - F(\boldsymbol{\omega})\nu_0 \tag{5}$$

Then, finally, the problem may be formulated as that of finding a vector ν_0 so as to guarantee the synthesis of a vector $\boldsymbol{\omega}$ satisfying Eq. (5) which minimizes functional (4). We can thus solve the problem of choosing the optimum structure by determining the maximum component of the state probability vector [1]

$$P(t) = \int_{-\infty}^{\infty} \boldsymbol{\omega}(\mathbf{Y},t)d\mathbf{Y} \tag{6}$$

We will now use the method of dynamic programming, according to which, when an optimum control is sought in the class of piecewise-continuous functions with values in an open domain ν_*, the problem reduces to solving the following functional equation

$$\min_{\nu\in\nu_*}\left\{\frac{dV}{dt}+W\right\}=0 \tag{7}$$

subject to the final condition $V(t_k)=0$, where V is the optimum functional, which depends parametrically on time $t\in T$ and is defined on the set of vector-valued functions ω satisfying Eq. (5).

For linear systems, the functional V is sought as an integral quadratic form [2]

$$V=\int_{Y_*}\omega^T(\mathbf{Y},t)a(\mathbf{Y},t)\omega(\mathbf{Y},t)d\mathbf{Y}$$

where a is an $s\times s$ matrix. Hence, taking into account that ω satisfies Eq. (5), we obtain an expression for the subsequent determination of the optimum ν_0^*

$$\frac{dV}{dt}+W=\int_{Y_*}\left\{\omega^T\frac{\partial a}{\partial t}\omega+\omega^T(a^T+a)(L(\omega)-F(\omega)\nu_0)+\Phi(I_s\omega)+\nu_0^T\nu_0\right\}d\mathbf{Y} \tag{8}$$

Analysis of this expression shows that the determination of the vector ν_0^* by solving the functional equation (7) reduces to a classical problem: it is required to find a vector-valued function which minimizes the definite integral (8). The required vector-valued function $\nu_0^*(\mathbf{Y},t)$ is also required to satisfy the system of Euler equations, whence it follows that

$$\nu_0^*=\tfrac{1}{2}F^T(\omega)(a^T+a)\omega$$

Substituting the optimum state transition law ν_0^* thus found into (5), one can write down the equation for the optimum vector ω in the sense of (4)

$$\frac{\partial\omega}{\partial t}=L(\omega)-\frac{1}{2}F^T(\omega)(a^T+a)\omega \tag{9}$$

Integration of this equation completes the solution of the problem of choosing the optimum structure by determining the maximum component of the state probability vector (6).

The equations necessary to determine the matrix-valued function $a(\mathbf{Y},t)$ in (9) follow from the condition

$$(dV/dt+W)_{\nu_0=\nu_0^*}=0$$

after substituting the vector ν_0^* into (8)

$$\frac{\partial a}{\partial t}=\frac{1}{s}(a^T+a)L(\omega)\omega_0^T-\frac{1}{s^2}\omega_0\omega_0^T\Phi(I_s\omega)+\frac{1}{4}(a^T+a)F(\omega)F^T(\omega)(a^T+a) \tag{10}$$

where $\omega_0=\|\,1/\omega^{(1)},\ldots,1/\omega^{(s)}\,\|^T$ is an auxiliary vector, introduced to facilitate the analysis and to simplify the form of Eq. (10).

Simultaneous solution of system (9), (10) subject to the boundary conditions

$$\omega(\mathbf{Y},t_0)=\omega_0,\quad V(Y,t_k)=0$$

essentially exhausts the theoretical solution of the problem formulated at the outset.

If the vector ν_0^* is constructed on the assumption that it is independent of $\mathbf{Y}$, the condition that the functional equation (8) should be minimized implies an integral dependence of ν_0^* on ω

$$\nu_0^*=(2\Delta)^{-1}\int_{Y_*}F^T(\omega)(a^T+a)\omega d\mathbf{Y},\quad \Delta=\int_{Y_*}d\mathbf{Y}$$

which, after substitution into (5) and (8), yields a system of integro-partial-differential equations (unlike (9), (10), which are only differential equations), the solution of which is considerably more complicated than in the first case.

Despite the fact that the theoretical solution of the problem obtained implies the possibility, in

principle, of an optimum choice of structure for the process Y, the practical solution of the boundary-value problem by direct consideration of the adjoint system of partial differential equations (9) and (10) is extremely difficult. Without dwelling on the numerous possible approximate methods for solving that problem, based on a compromise between the necessary precision and the volume of computational resources, we will now consider a method using expansion of the functions a and ω in series with respect to a certain system of orthonormal functions $\varphi = \| \varphi_1, \ldots, \varphi_N \|^T$ of a vector argument

$$a(\mathbf{Y}, t) = B(t)\varphi(\mathbf{Y}), \quad \omega(\mathbf{Y}, t) = A(t)\varphi(\mathbf{Y})$$

where $A(t)$ and $B(t)$ are ordinary and partitioned matrices of the expansion coefficients, which are determined as the solution proceeds. Then the problem reduces to a two-point boundary-value problem: integration of the matrix integro-differential equation

$$\frac{\partial A}{\partial t} = \int_{\mathbf{Y}} \left\{ L(A\varphi)\varphi^T - \frac{1}{2} F((A\varphi)F^T(A\varphi)(\varphi^T B^T + B\varphi)(A\varphi)\varphi^T) \right\} d\mathbf{Y}$$

The solution of this problem turns out to be much simpler and may be achieved by various traditional techniques: shooting, invariant imbedding, etc. A special feature in the practical solution of the problem in this case is that no rigid conditions are imposed on its precision, since the choice of structure depends only on the index of the maximum component of the vector

$$P(t) = A(t) \int_{\mathbf{Y}} \varphi(\mathbf{Y}) d\mathbf{Y}$$

and not on its value.

To illustrate the possibility of using the above approach, we consider the following example. A non-linear stochastic process with random structure is described by the equation

$$\dot{\mathbf{y}} = f^{(l)}(\mathbf{y}, t) + \mathbf{v}_t, \quad l = 1, 2$$
$$f^{(1)}(\mathbf{y}, t) = -\mathbf{y}^2, \quad f^{(2)}(\mathbf{y}, t) = -\mathbf{y} + 0.01\mathbf{y}^3$$

where $\mathbf{v}_t$ is normalized Gaussian white noise.

It is required to choose the structure of the process y so as to maximize the probability of its existence within given limits $\mathbf{y}_* = [y_{\min} = -0.8; y_{\max} = 0.9]$ over the time interval $T = [0, 300]$ s; that is, the minimizing criterion has the form

$$J = \int_{T} \int_{\mathbf{y}_*} \{-\rho(\mathbf{y}, t) + \nu_0^T(\mathbf{y}, t)\nu_0(\mathbf{y}, t)\} d\mathbf{y} dt, \quad \rho(\mathbf{y}, t) = \omega^{(1)}(\mathbf{y}, t) + \omega^{(2)}(\mathbf{y}, t)$$

where

$$\frac{\partial \omega}{\partial t} = L(\omega) - F(\omega)\nu_0, \quad \omega = \begin{Vmatrix} \omega^{(1)} \\ \omega^{(2)} \end{Vmatrix}, \quad \nu_0 = \begin{Vmatrix} \mathbf{v}_{12} \\ \mathbf{v}_{21} \end{Vmatrix}$$

$$L(\omega) = \begin{Vmatrix} \dfrac{\partial}{\partial \mathbf{y}}(\mathbf{y}^2\omega^{(1)}) + \dfrac{1}{2}\dfrac{\partial^2 \omega^{(1)}}{\partial \mathbf{y}^2} \\ \dfrac{\partial}{\partial \mathbf{y}}[(\mathbf{y} - 0{,}01\mathbf{y}^3)\omega^{(2)}] + \dfrac{1}{2}\dfrac{\partial^2 \omega^{(2)}}{\partial \mathbf{y}^2} \end{Vmatrix}, \quad F(\omega) = \begin{Vmatrix} \omega^{(1)} & -\omega^{(2)} \\ -\omega^{(1)} & \omega^{(2)} \end{Vmatrix}$$

Equations (9) and (10) for the optimal vector and adjoint matrix function in this case are

$$\frac{\partial \omega}{\partial t} = L(\omega) - \frac{1}{2} \begin{Vmatrix} \Omega_0 & -\Omega_0 \\ -\Omega_0 & \Omega_0 \end{Vmatrix} (a^T + a)\omega$$

$$\frac{\partial a}{\partial t} = \frac{1}{2}(a^T + a)L(\omega) \begin{Vmatrix} \dfrac{1}{\omega^{(1)}} & \dfrac{1}{\omega^{(2)}} \end{Vmatrix} + \frac{1}{4} \begin{Vmatrix} \dfrac{1}{\omega^{(1)}}\left(1 + \dfrac{\omega^{(2)}}{\omega^{(1)}}\right) & \dfrac{1}{\omega^{(1)}} + \dfrac{1}{\omega^{(2)}} \\ \dfrac{1}{\omega^{(1)}} + \dfrac{1}{\omega^{(2)}} & \dfrac{1}{\omega^{(2)}}\left(1 + \dfrac{\omega^{(1)}}{\omega^{(2)}}\right) \end{Vmatrix} + \frac{1}{4}(a^T + a) \begin{Vmatrix} \Omega_0 & -\Omega_0 \\ -\Omega_0 & \Omega_0 \end{Vmatrix} (a^T + a)$$

This problem may be solved by approximating the functions ω and a by Fourier series in the interval $[-5, 5]$, accurate to four terms of the expansions, and integrating the system of equations obtained for the expansion coefficients, using the invariant imbedding method [3] in the time interval [0, 300] s. On completing the integration, having computed approximate values of the functions $\omega^{(1)}$ and $\omega^{(2)}$, it was found that the indices of the structures, chosen subject to the requirement that the probability of the state at the time in question should be a maximum, were distributed in time as follows: in the intervals [0, 87] s and [115, 300] s—the second structure; in the intervals [0, 87] s and [115, 300] s—the second structure; in the intervals [0, 87] s and [115, 300] s—the second structure; in the interval [87, 115] s—the first structure. At the same time, the system of equations of the densities $\omega^{(1)}$ and $\omega^{(2)}$ for the traditional case of an uncontrollable change of state with unit intensity [1] was integrated, and it was established that in the latter case the value of the minimized criterion J was greater by a factor of 1.47 than in optimal control by the choice of structure.

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